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DETERMINATION OF THERMAL CONDUCTIVITY IN A NON-LINEAR HEAT CONDUCTION PROBLEM WITHOUT INTERIOR MEASUREMENTS

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ABSTRACT

This paper presents an integration approach to estimate temperature-dependent thermal conductivity in a transient non-linear heat conduction medium without internal measurements. The unknown thermal conductivity is assumed to vary linearly with respect to temperature. The integration approach to the inverse heat conduction problem requires the time-dependent temperature distribution that is not given *a priori*. For a one-dimensional heat conduction medium with a heated and an insulated wall, this study approximates the spatial temperature distribution as a polynomial with unknown coefficients, which satisfy four known boundary data (two prescribed heat fluxes and two measured temperatures) and the energy conservation. The integral heat conduction equations are solved to determine the unknown coefficients. Some numerical examples are introduced to show the performance of the proposed approach.

INTRODUCTION

The inverse heat conduction problem (IHCP) has been received much attention from many investigators due to practical importance and mathematical interest [1-12]. The determination of thermal properties from temperature measurements is a typical IHCP. In this paper we present an approach to estimate the thermal conductivity. In most practical applications, the thermal conductivity is dependent on the temperature and it may be assumed to vary linearly with respect to temperature within the range of interest. Then, the present IHCP becomes an inverse problem to estimate the unknown coefficients of the functional form of thermal conductivity that is set to vary linearly with temperature. Namely, the present IHCP can be classified into a coefficient identification problem.

Many researchers have been interested in the IHCP to determine the temperature-dependent thermal conductivity under the condition that the heat capacity per unit volume is a known constant. A number of methods, consequently, have been developed to solve such IHCPs [3-10]. Most of them adopted the nonlinear optimization formulation to find a thermal property minimizing the difference between the measured and the calculated temperatures at pre-specified spatial and temporal points. For example, Huang et al. [5] used a conjugate gradient method with an adjoint equation and Yang [8] applied a sensitivity method after the approximation of the thermal conductivity as a linear combination of known functions with unknown coefficients. The solution procedure is usually comprised of the forward problem stage to find the temperature profile with the assumed thermal properties and the inverse problem stage to update the unknown thermal properties to minimize the aforementioned difference and these stages are repeated until convergence [3-8]. In the forward stage, solving the heat conduction equation in the form of a partial differential equation may be substituted with solving the algebraic energy balance equation derived on the basis of the integration approach [4,9]. In contrast to the above iterative methods, some investigators attempted non-iterative approaches with the aid of the Kirchhoff transformation [10] to linearize the nonlinear heat conduction equation or the direct integration approach [11].

One of major difficulties in solving these IHCPs is that, due to the ill-posed nature of the problem, small noise in the measured temperatures may cause the solution diverged or large numbers of iterations are needed for convergence. Hence, the iterative regularization to mitigate the ill-posedness should be required and it would be noteworthy to mention that this issue is intensively dealt with by Alifanov et al. [1].

For determining good initial guesses to start the inverse solutions prior to the iterative optimization procedure to estimate the thermal properties, Huang and Özişik [2,4] proposed a direct integration approach. With their initial guesses obtained within about 20% error, they could enhance the iterative IHCP solution performance. In order to estimate the temperature-dependent thermal conductivity in a transient heat conduction medium [9,11] and the time-dependent boundary heat flux [12], Kim et al. formulated an integral balance equation for the heat conduction with the approximation of the spatial temperature profile as a third-order polynomial with four unknown coefficients that could be expressed in terms of two heat fluxes imposed and two temperatures measured at both boundaries. Also, this direct integration approach was applied to estimate the temperature dependent thermal conductivity and volumetric heat capacity simultaneously [13], and successfully compared with the Levenberg-Marquardt method, one of conventional iterative methods, of Huang and Özişik [4].

In this paper, we employ the integral approach proposed by Kim et al. [9] to the estimation of the temperature-dependent thermal conductivity since it has the advantages over Huang and Özişik's method [2,4]. The former uses the data measured at the boundary only while in the latter interior sensors to measure internal temperatures were inevitable. Furthermore, the present method solves the algebraic equations derived from the integral heat conduction equation rather than the partial differential equation for the heat conduction.

According to numerical experiments for small Fourier numbers which may be the measure of the ratio of the temperature-wave penetration depth to the system dimension, however, the approximate spatial temperature distribution of the third-order polynomial used in the integral approach of Kim et al. [9] deviates from the true distribution, although for large Fourier numbers, e.g. for $Fo >\sim 1$, the third-order polynomial is a good approximation. In consequence, the previous approach should require long measuring time to keep the Fourier number sufficiently large. This works intends to improve the integral approach to extend its applicability to the situations with smaller Fourier numbers. We approximate the spatial temperature distribution as a fourth-order polynomial with unknown coefficients, which satisfy four known boundary data (two prescribed heat fluxes and two measured temperatures). Additional constraint is considered to determine five unknown coefficients from the energy conservation.

In illustration of the performance of the proposed method, we introduce several examples, in which the thermal conductivity varies linearly or quasi-linearly with temperature. Also, a statistical analysis is carried out to show the confidence bounds in the estimation with noisy data since the inverse solution tends to very sensitive to the measurement error.

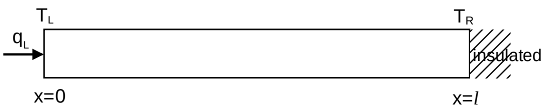

Fig. 1. Description of problem domain.

MATHEMATICAL MODEL

In a one-dimensional homogeneous heat conduction medium as shown in Figure 1, let us consider a problem to determine the temperature-dependent thermal conductivity, $k(T)$ where the heat capacity per unit volume, C , is a known constant. It is assumed that the domain is bounded on the left end by a heated wall and on the right by an insulated wall. The heat flux into the left wall is set to a constant, q_L . The transient heat conduction is governed by

$$C\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \quad x\in[0,l]\,,\ t\in(0,\infty) \qquad (1)$$

which is subject to the initial and boundary conditions

$$T(t=0,x)=0 \qquad (2)$$

$$q(t,x=0)=-k\left.\frac{\partial T}{\partial x}\right|_{x=0}=q_L \qquad (3a)$$

$$q(t,x=l)=-k\left.\frac{\partial T}{\partial x}\right|_{x=l}=0\,. \qquad (3b)$$

For the simplicity, the initial temperature distribution is assumed to be zero. It is assumed that two temperature sensors are installed at both ends. The measured temperatures will be time-dependent:

$$T(t,x=0)=T_L(t)\,,\ \ T(t,x=l)=T_R(t)\,. \qquad (4)$$

We consider the thermal conductivity that varies linearly with respect to temperature:

$$k(T)=k_0+k_1T\,. \qquad (5)$$

Now, the IHCP to estimate the temperature-dependent thermal properties is converted into a parameter identification problem to determine the coefficients k_0 and k_1 only with the boundary information from Eqs. (3) and (4). We apply the integral approach for the estimation of $k(T)$.

In order to apply the direct integration approach that does not solve the partial differential equation for the heat conduction, Eq. (1), to the present inverse analysis, Eq. (1) is integrated with respect to spatial and time coordinates:

$$\int_0^l\int_0^t w(x)C\frac{\partial T}{\partial t}dtdx=\int_0^t\int_0^l w(x)\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]dxdt \qquad (6)$$

where $w(x)$ is a pre-determined weighting function. If the temperature distribution is known, Eq. (6) constructs a relation the unknown coefficients k_0 and k_1 should satisfy. Or, if the temperature distribution can be approximated in terms of the unknown coefficients only without introducing additional unknowns, Eq. (6) can still describe the relationship between known parameters and the unknown k_0 and k_1.

Recalling the initial and boundary conditions considered in this study, one can expect that the temperature should be monotonically decreasing or increasing as the spatial coordinate according to the sign of heat flux, q_L. Hence, we can simply approximate the temperature distribution as a polynomial with time dependent coefficients:

$$T(t,x)\cong\sum_{n=0}^{N}a_n(t)\left(\frac{x}{l}\right)^n\,. \qquad (7)$$

where N is the order of the approximated temperature distribution. Since we have two applied boundary heat fluxes and two measured boundary temperatures, N may be set to be 3, which was used by Kim et al. for the estimation of the temperature-dependent thermal conductivity [9]. The coefficients will be

$$a_0=T_L\,, \qquad (8a)$$

$$a_1=-T_L^*\,, \qquad (8b)$$

$$a_2=2T_L^*-3(T_L-T_R)\,, \qquad (8c)$$

$$a_3=-T_L^*+2(T_L-T_R) \qquad (8d)$$

where

$$T_L^*\equiv\frac{q_Ll}{k_L}\,. \qquad (9)$$

In order to appreciate the approximation of a third-order polynomial, consider a heat conduction problem with constant thermal properties. The heat conduction equation will be rewritten as

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial\xi^2} \qquad \xi\in[0,1]\,,\ \tau\in(0,\infty) \qquad (10)$$

where $\xi=x/l$ is the dimensionless length, $\tau=k_0t/C_0l$ so called Fourier number, and $\theta=k_0T/q_Ll$ the dimensionless temperature. The subscript '0' denotes reference value. As can be seen in Fig. 2, the third-order approximation seems to be quite good for large Fourier numbers, e.g. τ >0.1, although for small Fourier numbers the approximate deviates from the true distribution. As the Fourier number is smaller, the deviation will be more pronounced. It is likely, of course, that for non-linear heat conduction problems the deviation may be magnified. Hence, it is expected that the previous integral approach employing the third-order approximation should require a slow heating (small q_L and long measuring time) to keep the Fourier number sufficiently large.

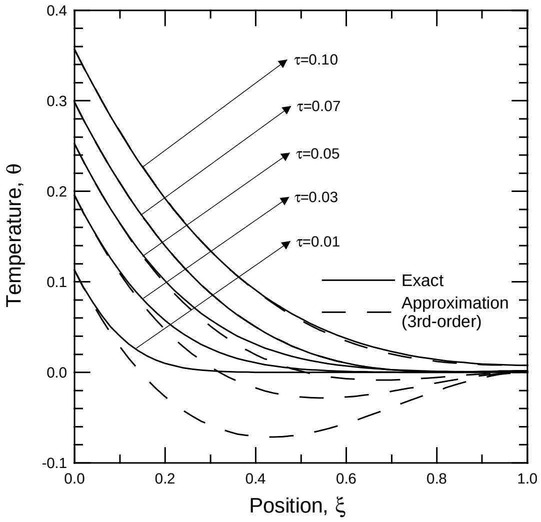

Fig. 2. Approximated tmeprature distribution with a third-order polynomial for a linear heat conduction problem.

Recalling the energy balance, in fact, we can use one more constraint to approximate the temperature distribution. The energy balance corresponds to the case of $w(x)=1$ in Eq. (6):

$$\int_0^l\int_0^t C\frac{\partial T}{\partial t}dtdx=\int_0^t\int_0^l\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]dxdt \qquad (11a)$$

or from initial and boundary conditions we have

$$C\int_0^l T\,dx=q_L t \qquad (11b)$$

With this additional constraint, we can approximate the spatial temperature distribution as a fourth-order polynomial ($N=4$) and the coefficients will read

$$a_0=T_L, \qquad (12a)$$

$$a_1=-T_L^*, \qquad (12b)$$

$$a_2=\left(\frac{9}{2}+\frac{30k_L}{k_0}\frac{C_0}{C}\tau\right)T_L^*-18T_L-12T_R, \qquad (12c)$$

$$a_3=-\left(6+\frac{60k_L}{k_0}\frac{C_0}{C}\tau\right)T_L^*+32T_L+28T_R, \qquad (12d)$$

$$a_4=\left(\frac{5}{2}+\frac{30k_L}{k_0}\frac{C_0}{C}\tau\right)T_L^*-15T_L-15T_R. \qquad (12e)$$

Figure 3 illustrates the approximated temperature distribution with a fourth-order polynomial and the comparison shows the improvement in the deviation from the true distribution.

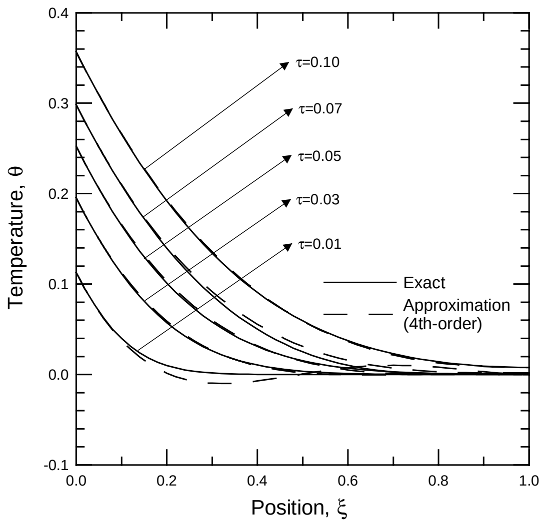

Fig. 3. Approximated tmeprature distribution with a fourth-order polynomial for a linear heat conduction problem.

Hence, we can expect that the fourth-order approximation may generate more favorable results in the estimation of the unknown coefficients, that is the temperature-dependent thermal conductivity, especially for smaller Fourier numbers.

In order to construct a functional the unknown coefficient should satisfy, we set the weighting function to $w(x)=1-x$ and each side of Eq. (6) becomes

$$LHS(t) = \int_0^l \int_0^t (1-x)C\frac{\partial T}{\partial t}dtdx = C\int_0^l (1-x)Tdx\,, \tag{13a}$$

$$RHS(t) = \int_0^t \int_0^l (1-x)\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]dxdt$$
$$= q_L lt - \int_0^t \frac{1}{2}\left[(k_L + k_0)T_L - (k_R + k_0)T_R\right]dt \tag{13b}$$

respectively. In this, the weighting function of $w(x) = 1 - x$ has been chosen since the largest temperature increase occurs at the heated wall. Due to the linear approximation of thermal properties, we have

$$k\frac{\partial T}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left[(k + k_0)T\right]. \tag{14}$$

When obtaining Eq. (13b), the above relation is used.

With the approximated temperature distributions Eqs. (8) and (12), the weighted integrals of the temperature distribution will be

$$\int_0^l Tdx = = l\left[-\frac{T_L^*}{12} + \frac{1}{2}\left(T_L + T_R\right)\right] \tag{15a}$$

$$\int_0^l xTdx = l^2\left[-\frac{T_L^*}{30} + \frac{1}{20}(3T_L + 7T_R)\right] \tag{15b}$$

for a third-order approximation, and

$$\int_0^l Tdx = l\, T_L^* \frac{k_L}{k_0}\frac{C_0}{C}\tau \tag{15c}$$

$$\int_0^l xTdx = l^2\left[\left(\frac{1}{120} + \frac{1}{2}\frac{k_L}{k_0}\frac{C_0}{C}\tau\right)T_L^* - \frac{T_L}{10} + \frac{T_R}{10}\right] \tag{15d}$$

for a forth-order approximation.

In practice, boundary temperatures will be measured discretely at predetermined temporal coordinates t_m, $m = 1,2,\cdots,M$. That is, M is number of measurements. Now, the present inverse problem turns into a problem to find the coefficients k_0 and k_1 minimizing the following functional:

$$\Phi = \sum_{m=1}^{M} \frac{1}{2}\left[LHS(t_m) - RHS(t_m)\right]^2 . \tag{16}$$

To find a coefficient vector $\alpha = \left[k_0\ k_1\right]^T$ which minimizes Φ, we use usual Newton-Raphson method. It should be noted that according to our numerical experiments the present optimization problem doen not require any regularization to mitigate the ill-posedness that is commonly encountered in inverse solutions.

NUMERICAL EXPERIMENTS

For the evaluation of the proposed algorithm, we consider several examples, in which the thermal conductivity varies with the temperature linearly or quasi-linearly. First, two examples of linearly varying thermal conductivity are examined. The thickness of the specimen is set to l =0.03m.

Example 1:
C =4,000 [kJ/m^3-℃]
$k(T) = 75.5 - 0.07T$ [W/m-℃]

Example 2:
C =4,000 [kJ/m^3-℃]
$k(T) = 0.07(T - 200) + 75.5$ [W/m-℃]

In this, the temperature has a unit of ℃. The first

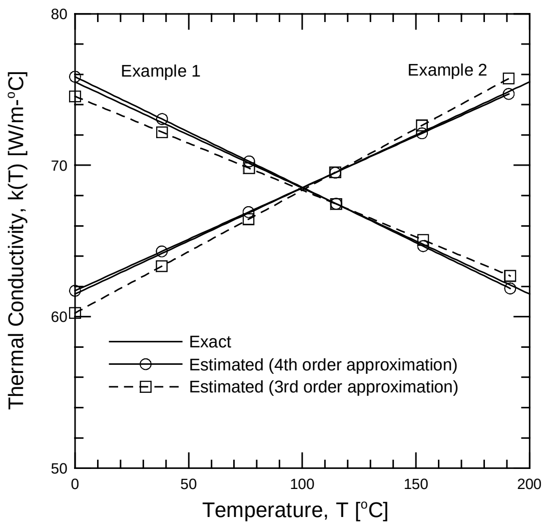

Fig. 4. Estimated thermal conductivities for Examples 1 and 2.

example, in fact, corresponds to iron specimen and the second is chosen arbitrarily to simulate a thermal conductivity increasing with temperature. The total temperature measuring time, t_{end} , is taken as 900s. It is assumed that 100 temperature readings with an even measurement time interval are performed. The heat flux at the left boundary is assumed to be $q_L = 25{,}000$ W/m^2. The reconstructed thermal conductivity variations are plotted in Fig. 4, which shows an excellent performance of the present integral approach. It also means that both temperature distributions, namely third- and fourth-order polynomials, can approximate the true distribution quite reasonably. Although the result with the fourth-order approximation is more favorable, the difference does not seem to be significant.

Now, let's consider the third example in which the thermal conductivity somewhat deviates from the linear dependency on temperature. This example may be more probable in reality.

Example 3:

C =1,740 [kJ/m^3-K]

$k(T) = 45{,}100T^{-1.2}$ [W/m-K]

In this, the temperature is in K. The specimen of thickness l =0.05m is initially in thermal equilibrium at 300K, and the left wall is suddenly heated by a constant heat flux of q_L =4,000W/m^2, while the right wall is insulated. Total measurement time is 5,000s and the specimen will be heated up to about 525K. The temperature readings at both ends are taken every 2.5s. These thermal properties are for gallium arsenide. The ineluctability of the measurement error could make the inverse solution diverged. Hence, thestatistical analysis is important to validate the performance as well as to determine the accuracy of the developed inverse algorithm. In the numerical experimentation, we impose a Gaussian noise on the measured temperature

$$T_{measured} = T_{exact} + \omega\sigma \qquad (17)$$

where σ is the standard deviation of the measured temperatures and ω is a random number which lies within the specified confidence bounds. If we use 99% confidence bounds, the random number ranges -2.567< ω <2.567;

$$\textit{Probability} \; (T_{measured} - 2.576\sigma < T_{exact} < T_{measured} + 2.576\sigma) \cong 99\% \qquad (18)$$

The standard deviation is set to σ =0.5℃. To obtain statistical data 100 numerical experiments are conducted. The comparison between the estimated and the exact thermal conductivities is made in Fig. 5, which shows an excellent agreement. The estimated thermal conductivity with the third-order polynomial approximation shown in Fig. 5 is obtained under error-free condition.

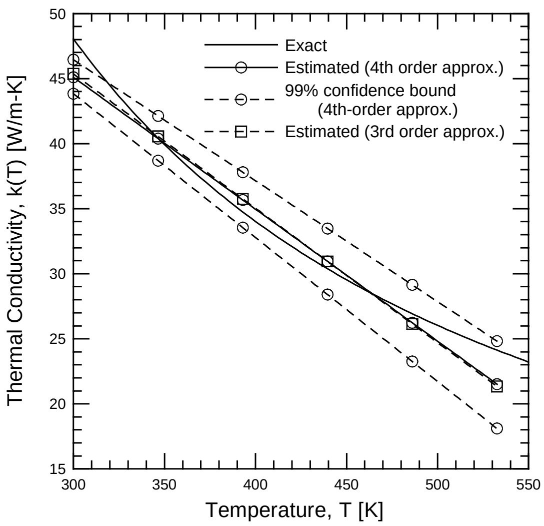

Fig. 5. Estimated thermal conductivities for Example 3 (σ =0.5℃).

Considering the fitness of the approximated temperature distribution to the exact distribution and recalling the comparisons made in Figs. 2 and 3, it would be illustrative to examine the Fourier number based on the reference thermal properties of each example, which is defined as

$$Fo = \frac{k_0 t}{C_0 l^2} \qquad (25)$$

In this, the subscript '0' means the reference value and for the above three examples we set the reference values as k_0 = 70W/m-K and C_0 =4,000 kJ/m^3-K for Examples 1 and 2 and

k_0 = 30W/m-K and C_0 =2,000 kJ/m^3-K for Example 3.

For the first two examples, t_{end} =900s corresponds to Fourier number of 17. In Example 3, the Fourier number for t_{end} =5,000s is 30. All the examples have sufficiently large Fourier numbers, which will be favorable in fitting the approximated distribution to the exact one. In order to evaluate the effect of the Fourier number on performance of the present approach, hence, we introduce other experimental conditions with different heat flux and different measuring time for the same specimen of Example 3.

Case 1: q_L =40kW/m^2, t_{end} =500s, Fo=3
Case 2: q_L =80kW/m^2, t_{end} =250s, Fo=1.5
Case 3: q_L =200kW/m^2, t_{end} =100s, Fo=0.6

As the Fourier number decreases, the predictability of the integral approach based on the third-order polynomial approximation of the temperature distribution is degraded as shown in Fig. 6. As for Case 3 of Fo=0.6, the estimated thermal conductivity is far from the true one, while the present approach with the fourth-order polynomial approximation can estimate the unknown coefficients quite reasonably. From the results of Case 3 given in Fig. 6c, we can find the deviation of the estimated thermal conductivity from the exact one is magnified in the region of lower temperatures, which corresponds to earlier time (i.e. smaller Fourier number). Such deviation can be explained by the fact that for smaller Fourier numbers the approximation of the temperature distribution may show poor agreements with the exact temperature distribution, as shown in Fig. 3. It would be intersting that the error bound narrows as the Fourier number decreses. The errors of the estimated thermal conductivity are summarized in Fig. 7. In this, the error is defined as:

$$Error = \frac{\int_{T_{\min}}^{T_{\max}} \left| k_{estimated}(T) - k_{exact}(T) \right| dT}{\int_{T_{\min}}^{T_{\max}} k_{exact}(T) dT} \quad (26)$$

CONCLUSIONS

An integral approach to estimate temperature-dependent thermal conductivity is proposed and examined for a one-dimensional non-linear heat

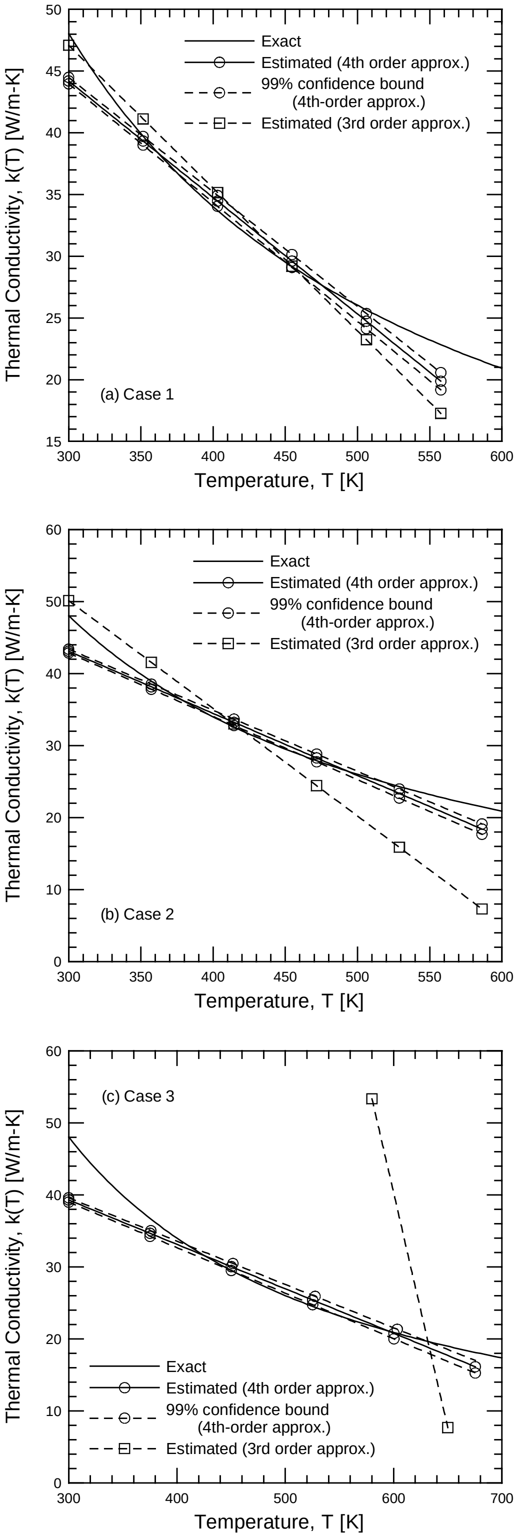

Fig. 6. Effect of Fourier number on the estimated results (σ =0.5℃).

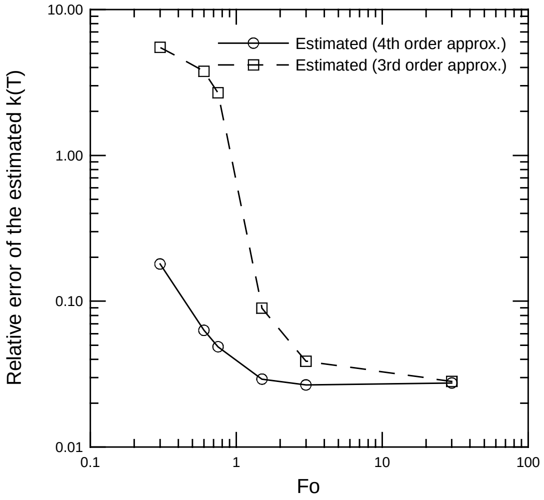

Fig. 7. Relative errors of estimated thermal conductivity as Fourier number.

conduction medium. The spatial temperature distribution is approximated as a third-order and a fourth-order polynomial. The unknown coefficients of the third-order polynomial are expressed in terms of the prescribed boundary heat fluxes and the measured boundary, and for the fourth-order an additional constraint of the energy balance is considered. With using the fourth-order polynomial approximation, we can improve the performance of the present integral approach.

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REFERENCES

1. O. M. Alifanov, E. A. Artyukhin and S. V. Rumyantsev, Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Heat Transfer Problems, Bell House, Inc., New York, 1995, pp. 42~93.

2. C. H. Huang and M. N. Özişik, A direct integration approach for simultaneously estimating spatially varying thermal conductivity and heat capacity, *Int. J. Heat Fluid Flow*, 11, 262 (1990).

3. T. Jurkowski, Y. Jarny and D. Delaunay, Estimation of thermal conductivity of thermoplastics under moulding conditions: an apparatus and an inverse algorithm, *Int. J. Heat Mass Transfer* **40**, 4169 (1997).

4. C. H. Huang and M. N. Özişik, Direct integration approach for simultaneously estimating temperature dependent thermal conductivity and heat capacity, *Numer. Heat Transfer – A*, **20**, 95 (1991).

5. C. H. Huang, J. Y. Yan and H. T. Chen, Function estimation in predicting temperature-dependent thermal conductivity without internal measurements, *J. Thermophys. Heat Transfer*, **9**, 667 (1995).

6. H. T. Chen, J. Y. Lin, C. H. Wu and C. H. Huang, Numerical algorithm for estimating temperature-dependent thermal conductivity, *Numer. Heat Transfer – B*, **29**, 509 (1996).

7. H. T. Chen and J. Y. Lin, Simultaneous estimations of temperature-dependent thermal conductivity and heat capacity, *Int. J. Heat Mass Transfer*, **41**, 2237 (1998).

8. C. Y. Yang, Estimation of the temperature-dependent thermal conductivity in inverse heat conduction problems, *Appl. Math. Model.*, **23**, 469 (1999).

9. S. Kim, M. C. Kim and K. Y. Kim, An integral approach to the inverse estimation of temperature-dependent thermal conductivity without internal measurements, *Int. Comm. Heat Mass Transfer*, **29**, 107 (2002).

10. D. Lesnic, L. Elliott, D. B. Ingham, A note of the determination of the thermal properties of a material in a transient nonlinear heat conduction problem, *Int. Comm. Heat Mass Transfer*, **22**, 475 (1995).

11. S. Kim, M. C. Kim and K. Y. Kim, Non-iterative estimation of temperature-dependent thermal conductivity without internal measurements, *Int. J. Heat Mass Transfer*, **46**, 1801 (2003).

12. S. Kim, M. C. Kim and K. Y. Kim, A direct integration approach for the estimation of time-dependent boundary heat flux, *KSME Int. J.*, **16**, 1320 (2002).

13. S. Kim, B. J. Chung, M. C. Kim, K. Y. Kim, Inverse estimation of temperature-dependent thermal conductivity and heat capacity per unit volume with the direct integration approach, *Numer. Heat Transfer – A*, **44**, 521 (2003).